J. Fluid Mech. (1967), *vol.* 27, *part* 3, *pp.* 513–539
Printed in Great Britain

The critical layer for internal gravity waves in a shear flow

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(Received 25 April 1966)

Internal gravity waves of small amplitude propagate in a Boussinesq inviscid, adiabatic liquid in which the mean horizontal velocity $U(z)$ depends on height z only. If the Richardson number R is everywhere larger than $\frac{1}{4}$, the waves are attenuated by a factor $\exp\{-2\pi(R-\frac{1}{4})^{\frac{1}{2}}\}$ as they pass through a critical level at which U is equal to the horizontal phase speed, and momentum is transferred to the mean flow there. This effect is considered in relation to lee waves in the airflow over a mountain, and in relation to transient localized disturbances. It is significant in considering the propagation of gravity waves from the troposphere to the ionosphere, and possibly in transferring horizontal momentum into the deep ocean without substantial mixing.

1. Introduction

Many authors have studied theoretically small adiabatic perturbations to a shear flow in a stratified inviscid fluid, in which the mean motion is uniform in horizontal planes but varies with height z. Such perturbations may be of very different types, ranging through sound waves and through motions depending crucially on the vorticity in the basic flow to what are essentially gravity waves.

If the fluid density varies greatly across the flow the differences in inertia per unit volume may be of great dynamical importance, but if the total range of density is small the predominant influence of its variations may be through changes in the weight of fluid per unit volume. Under such circumstances, a self-consistent approach, generally known as the Boussinesq approximation, is to regard the fluid as uniform and incompressible, except for the introduction of a buoyancy force in the vertical direction, equal to the gravitational acceleration g times the fluctuations of the density at each point and time from its mean value there $\bar{\rho}(z)$. The stratification of the mean flow may then be described in terms of a single parameter which may vary with z, the Brunt–Väisälä frequency N, defined by

$$N^2(z) = -\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz}. \tag{1.1}$$

This is the appropriate definition for a liquid. In a compressible atmosphere, the same approximations may still be applied to disturbances of which the vertical scale is small compared to the density scale height of the atmosphere, provided that $\bar{\rho}$ in equation (1.1) is replaced by the mean potential density, i.e. by the density that the air at height z would have if it were reduced adiabatically to a standard pressure. Sound waves are thereby excluded.

In this paper the Boussinesq approximation is a convenient framework in which to develop concepts which depend essentially on the buoyancy forces and their interplay with the shear. These concepts may probably be extended into a wider context, but for the present the approximation is adopted without comment.

The importance of vertical variations in the mean horizontal velocity, which has Cartesian components $(U(z), V(z), 0)$, is measured locally by the Richardson number

$$R = \frac{N^2}{U_z^2 + V_z^2}, \tag{1.2}$$

where the variable z in suffix position denotes differentiation. It was shown by Miles (1961) and Howard (1961) that, if R is everywhere greater than $\frac{1}{4}$, small disturbances to the mean flow will show no tendency to grow spontaneously with time in an exponential manner; i.e. the flow is stable to disturbances of small magnitude. We shall confine our attention to such stable situations.

If the basic velocity (U, V) vanishes, small disturbances take the form of internal gravity waves, in which there is an oscillatory interchange between disturbance kinetic energy and gravitational potential energy associated with deformation of the surfaces of constant density. The properties of such waves are described succinctly by Chandrasekhar (1961, p. 85) and in more generality by Eckart (1960). If U, V vary with z, the disturbance is modified by the shear, but, if the Richardson number is larger than $\frac{1}{4}$, the energy interchange is still of the same type as internal gravity waves.

Such internal gravity waves, modified to a greater or less extent by shear, are common in the terrestrial atmosphere, and probably also in many stellar atmospheres. For a review of reported cases see Bretherton (1966). They may be travelling relative to the surface of the earth, having been caused by some transient irregularity such as a thermal, or they may be stationary, appearing as lee waves behind a mountain range or other obstacle. They are also found almost everywhere in the oceans (La Fond 1962).

In this paper we draw attention to a mechanism whereby internal gravity waves, once generated, may be reabsorbed by the mean flow without necessarily invoking turbulence or other dissipative processes. It arises for a sinusoidal wave with horizontal phase velocity c, at a level z_c at which the component of the basic flow parallel to the horizontal wave-number is equal to c. At such a level, if the basic velocity profile is such that it exists, the frequency of the wave relative to the surrounding fluid vanishes. It emerges that, as a wave propagates vertically through the critical level, it is strongly attenuated. The Reynolds stress, which is an appropriate measure of the magnitude of the wave, is reduced on the other side by a factor

$$\exp\{-2\pi(R_c - \tfrac{1}{4})^{\frac{1}{2}}\}, \tag{1.3}$$

where R_c is the effective Richardson number at the critical level based on the vertical gradient of the component of the basic flow parallel to the horizontal wave-number. R_c is always larger than or equal to R. If R_c is unity or larger this is a very substantial reduction (5×10^{-3} when $R_c = 1$, 6×10^{-6} when $R_c = 4$). The range $\frac{1}{4} < R < 1$ is rare in the atmosphere, except possibly below jet streams, and we shall give most of our attention to the case when R is moderately large (unity or larger). Viscosity and heat conduction are entirely ignored, so the absorption does not depend on their action. Extension of the analysis to include them is probably feasible but has not yet been done. By analogy with the critical layer in hydrodynamic stability theory for unstratified flows (Lin 1955, ch. 8) it may turn out that the change in Reynolds stress described by (1.3) is independent of their magnitude, provided only they are small.

Scorer (1949) investigated the stationary train of lee waves behind a two-dimensional mountain in a stably stratified air stream of which the mean horizontal velocity $U(z)$ normal to the mountain crest varies with height z. A stationary sinusoidal wave train is possible in which the vertical disturbance velocity has the form

$$w(x,z) = R[\hat{w}(k,z)\,e^{ikx}], \tag{1.4}$$

where

$$\hat{w}_{zz} + \left(\frac{N^2}{U^2} - \frac{U_{zz}}{U} - k^2\right)\hat{w} = 0. \tag{1.5}$$

This equation is based on the Boussinesq approximation and a linearization about the basic flow.

The original object of the present study was to clarify theoretically how lee waves are affected if the wind $U(z)$ reverses at some height z_c. At such a critical level equation (1.5) is singular. It is not obvious whether solutions for the regions above and below z_c may legitimately be joined to one another, or if so how. The present approach yields information about the time development of lee waves in such a basic flow, but the final steady state is beyond the scope of the theory. Information is gained about the dynamical processes leading to the development of large horizontal perturbation velocities around z_c but ultimately the linearization used becomes invalid, and the waves probably degenerate into turbulence near the critical level. Nevertheless, the principles of the analysis are also applicable in other circumstances which are not open to this objection.

The two-dimensional transient disturbance produced by temporary extraneous forces may also be represented as the superposition of a continuum of travelling sinusoidal waves

$$w = R\left[\frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^{+\infty} dc\,\hat{w}(k,z,c)\,e^{ik(x-ct)}\right]. \tag{1.6}$$

Each Fourier component has a well-defined horizontal wave-number k (assumed positive) and phase velocity c, and a vertical structure satisfying

$$w_{zz} + \left\{\frac{N^2}{(U-c)^2} - \frac{U_{zz}}{U-c} - k^2\right\} w = 0, \tag{1.7}$$

which is, not surprisingly, Scorer's equation for a stationary wave in a mean flow $U(z)-c$. Equation (1.7) was originally derived by Synge (1933). Again the problem arises of matching across the critical level, but consideration of an initial value problem shows that this should be resolved by reference to the complex c-plane, with the imaginary part $c_i > 0$. This procedure leads inevitably to the conclusion that the magnitude of the wave is different above and below, and if R is moderately large a disturbance arising on one side of the critical level is effectively confined to that side. If $R < \frac{1}{4}$ the matching is quite different, and it is sometimes possible, though not inevitable, for normal mode disturbances to grow spontaneously without exterior forcing. The basic flow is then unstable. We shall consider only $R > \frac{1}{4}$ and invoke the Miles–Howard (1961) theorem that such unstable waves do not exist.

Such a resolution presupposes that at a given instant of time the velocity field is known everywhere and the subsequent developments are being followed. Then a Laplace transformation in time is appropriate. The inverse transform is formally identical to a Fourier transform, except that the contour of integration lies in the complex c-plane circumventing any singularities (such as at $c = U(z)$) in a prescribed manner. If it is required that, at some time in the future, the velocity field shall take on a predetermined form, then the resolution with $c_i < 0$ is the appropriate one, and the matching conditions across z_c are different. Such a condition, however, violates our preconceptions about causality, and we shall consider only systems in which causes are prescribed, and events follow causes. An alternative procedure is to keep c strictly real, but to include heat conduction and viscosity in the problem. This leads to a sixth-order differential equation in $w(z)$. It has not been established whether this leads to the same conclusion as the initial value problem.

For a localized transient disturbance a continuous range of values of the wave-number k and phase velocity c must be considered, with the appropriate vertical structure $w(z)$ for each. The critical level will thus be different for each Fourier component (if it exists at all), and the singularity at z_c for each separate component is not manifest in the complete velocity field, and the linearization remains valid. Nevertheless, the qualitative effect of the critical levels on the disturbances as a whole is profound. The absorption takes place over a finite volume, but is none the less real for that.

This absorption follows from what is at first sight a technical detail in the mathematics. A more physical explanation has been given by Bretherton (1966), who examined the motion of wave packets of arbitrarily small vertical wavelength in a stratified shear flow under the assumption that the Richardson number was also arbitrarily large. These assumptions are necessary for the consistency of the concepts of wave packet and group velocity. It was shown that a wave packet, moving with the appropriate local group velocity, would approach the critical level for the dominant frequency and wave-number for the packet, but would not reach it in any finite time. It would thus be neither reflected nor transmitted, and effectively absorbed. As the wave approaches the critical level its horizontal wave-number and frequency relative to the ground remain constant, but its frequency relative to the surrounding fluid decreases to zero. This is associated with a decrease in the vertical wavelength and the disturbance velocity becomes more and more nearly horizontal. At the same time, all the components of the propagation velocity relative to the surrounding fluid become smaller and smaller.

In the present analysis the concept of group velocity is not invoked, and the Richardson number is finite. Some of the wave motion does penetrate the critical layer, but if the Richardson number is moderately large this amount is exponentially small. No approximations are made, other than the Boussinesq approximation and the linearization about the basic state. However, it is an unfortunate fact that, although the mathematical principles on which the present result is based are quite general and of wide application, it is difficult to find simple illustrations in which all the integrals involved can be evaluated in terms of elementary functions and the complete velocity field worked out and presented. Although some formal solutions are easily obtained, to comprehend them recourse must be had to methods for the asymptotic evaluation of integrals. These are difficult to present compactly together with a discussion of all the analytical subtleties, and they obscure the simplicity of the basic argument.

Accordingly, we present in §§2, 3 a general discussion of the motion in the neighbourhood of the critical level for a single sinusoidal component, together with evidence for the correct physical interpretation of various terms. In §4 we consider formally a specific example, the time development of a stationary train of waves above a corrugation in the lower boundary which is introduced suddenly at time $t = 0$, and in §5 we consider the flow field for this problem a long time later, although the details of the asymptotic analysis are somewhat sketchy. In §6 we consider some of the features of the flow after a general transitory disturbance has been induced in it and largely propagated away and reabsorbed. The characteristic pattern of critical-layer absorption is found for a continuous spectrum of frequencies but a single sinusoidal wavelength, and this is extended to a narrow band of wavelengths. These qualitative features are interpreted by considering the general disturbance as the superposition of a large number of wave packets. Finally, we calculate the additional momentum imparted to the mean flow by the critical-layer absorption of a localized disturbance. Once obtained, this can be immediately generalized to show the effect of a random distribution of disturbances.

Throughout, only two-dimensional perturbations to a unidirectional mean velocity field $U(z)$ will be considered. Fourier analysis of a three-dimensional disturbance in a wind whose direction changes with height is inevitably in terms of two-dimensional waves, each with its own (vector) horizontal wave-number, and each affected only by the mean flow parallel to the horizontal wave-number. Thus all inferences about the critical layer in two-dimensional motion are directly applicable in three dimensions, with the proviso that the effective Richardson number will depend on the orientation of the wave-number, but is always greater than or equal to that defined in (1.2).

In the interpretation given in this paper, stress is laid on wave motions which are propagating at an angle to the horizontal, either upwards or downwards, rather than on the oscillations which occur in the waveguide formed by a stratified fluid between two rigid horizontal bounding surfaces. In describing standing lee waves it is natural to look for circumstances in which wave energy is trapped in a waveguide near the surface of the earth by some natural lid at which total reflexion of an upward propagating wave takes place. This occurs, for example, if there is an overlying deep layer of homogeneous liquid ($N^2 = 0$; U, V constant) for it is apparent that equation (1.7) is there exponential in type. However, such total internal reflexion is exceptional, and usually some at least of the wave energy continues to propagate upwards (Eliassen & Palm 1960). The waveguide is then imperfect. Upward propagation may continue to great heights and is probably of great importance in the upper atmosphere (Hines 1963). Anyhow, it is normally physically instructive to distinguish as far as possible between the upward and the downward propagating components at each level. If there is a critical level it is imperative, for waveguide modes with real phase velocity do not exist. This follows because the Reynolds stress for the upward and downward components must then be everywhere equal and opposite, in order to satisfy the upper and lower boundary conditions, which is inconsistent with the matching conditions across z_c.

One geophysical application of these results is concerned with limitations on the propagation of gravity waves from the troposphere up to the ionosphere. This has already been discussed by Bretherton (1966). At a late stage in the preparation of the present manuscript the authors discovered that absorption at a critical layer has also been postulated simultaneously and independently by Hines & Reddy (1966), although the method of analysis is different, and some of the consequences for the upper atmosphere are discussed there.

Another possible application, which awaits quantitative evaluation, is in the oceans. Horizontal phase speeds for internal gravity waves there are generally less than 1 m/s, and considerably smaller in the higher waveguide modes. No direct measurements have been made of phase velocities, but it seems plausible from considering orders of magnitude of the ocean currents that critical levels will occur fairly frequently. It is not possible from existing data to estimate how much energy or momentum is present in modes with a critical level. The mechanism described here then implies the vertical transfer of horizontal mean momentum into the depths of the ocean, without significant mixing of fluid particles. In other words, it exposes the possibility of an effective vertical diffusivity for momentum which is very much larger than the diffusivity for salt or other tracers used in water mass analysis. If substantiated, this conclusion could have considerable implications in the theory of the general circulation of the oceans.

2. Sinusoidal waves near the critical layer

We make the following assumptions:

(*a*) the motion is two dimensional;

(*b*) it is inviscid and adiabatic;

(*c*) the Boussinesq approximation;

(*d*) the rotation of the earth may be neglected;

(*e*) the perturbation velocities (u, w) from the basic state $U(z)$ are so small that

$$\left| u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z} \right| \ll \left| \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} \right|$$

throughout the region of interest;

(*f*) the Richardson number

$$R = N^2/U_z^2 > \tfrac{1}{4} \quad \text{everywhere;}$$

(*g*) at each critical level $U_z > 0$.

It is of interest to inquire into the effects of relaxing assumptions (*b*) and (*d*), but this cannot be pursued here. Assumptions (*a*) and (*g*) are made for convenience of presentation. If $U_z < 0$ at the critical level, some signs must be altered in the matching conditions, and the interpretation of some terms must be reversed. The differences are tabulated at the end of §3. Assumption (*e*), on the other hand, is of critical importance to the analysis. For a pure sinusoidal wave it would seem inevitably to fail just at the critical level where $\partial/\partial t + U\,\partial/\partial x$ vanishes. However, every disturbance which is not completely steady or periodic in time must be represented as the integral over many sinusoidal components, each with a different critical level, and the importance of the non-linear effects must be measured by the effects of the operators in assumption (*e*) on the complete velocity field, not on the individual components. This importance may be assessed *a posteriori*, by making the linearization, computing the velocity field on that basis, and then substituting to obtain the magnitude of the non-linear terms. If the velocity field obtained by the linearized theory is everywhere finite and differentiable, then for sufficiently small amplitude disturbances the linearization is justified. A pure sinusoidal wave with real phase velocity always has a singularity at the critical level, but the disturbances considered here, which develop from a state of rest relative to the basic state, are all continuously differentiable at all finite times after their initiation. In the motion above a stationary corrugation on a boundary, discussed in §§4, 5, the non-linear terms ultimately become important in a neighbourhood of the critical level, however small the corrugation, but, in the disturbance produced by a transient stimulus, this is only so after the disturbance has been almost entirely absorbed, if at all.

It may be shown (Bretherton 1966) that the vertical velocity w satisfies the linearized equation

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 (w_{xx} + w_{zz}) - \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(U_{zz}w_x) + N^2 w_{xx} = 0. \tag{2.1}$$

It follows at once that a Fourier component with the form (1.6) has a vertical structure satisfying equation (1.7). For a uniform stream with U independent of z, this reduces precisely to the governing equation for internal gravity waves, the gravitational restoring force being measured by N^2. In a shear flow, the vertical variations of mean vorticity may also provide a restoring force, measured by $-(U-c)U_{zz}$. The critical level z_c for the component, if it exists, is defined by

$$U(z_c) - c_r = 0, \tag{2.2}$$

where

$$c = c_r + ic_i. \tag{2.3}$$

If c is real ($c_i = 0$), equation (1.7) is singular there.

The mathematics necessary to connect solutions of equation (1.7) on the two sides of the singularity is given by Miles (1961). Our main concern in this paper is with the interpretation of this analysis, so it is necessary to restate part of it. We shall assume that c_i is small, specifically

$$|U_{zz}|\, U_z^{-2} c_i \ll 1. \tag{2.4}$$

Then we may use the method of Frobenius to obtain expansions in power series of the two independent solutions. Near the critical level the complete solution has the form

$$W = A\left(z - z_c - \frac{ic_i}{U_z}\right)^{\frac{1}{2}+i\mu} + B\left(z - z_c - \frac{ic_i}{U_z}\right)^{\frac{1}{2}-i\mu}, \tag{2.5}$$

where A and B are arbitrary constants and

$$\mu = (R - \tfrac{1}{4})^{\frac{1}{2}} > 0. \tag{2.6}$$

If $c_i = 0$, there is a branch point at $z = z_c$ in both solutions. The inclusion of further terms in the power series does not modify the structure near the branch point; it multiplies each solution by a function of $z - z_c$ and of c_i which is analytic in each variable in a neighbourhood of the origin and tends to unity there.

If $c_i > 0$, then, as $z - z_c$ decreases from positive values which are large compared with c_i/U_z to negative values, the argument of $z - z_c - ic_i/U_z$ changes continuously from 0 to $-\pi$. If, for the sake of definiteness, we fix the branch of the complex powers in (2.5) by taking

$$(z - z_c)^{\frac{1}{2}+i\mu} = |z - z_c|^{\frac{1}{2}}\, e^{i\mu|z - z_c|} \quad \text{if} \quad z > z_c, \tag{2.7}$$

then it follows that the correct interpretation is

$$\left.\begin{aligned} (z - z_c)^{\frac{1}{2}+i\mu} &= -i\, e^{\mu\pi}\, |z - z_c|^{\frac{1}{2}}\, e^{-i\mu|z - z_c|} \\ (z - z_c)^{\frac{1}{2}-i\mu} &= -i\, e^{-\mu\pi}\, |z - z_c|^{\frac{1}{2}}\, e^{i\mu|z - z_c|} \end{aligned}\right\} \quad \text{if} \quad z < z_c. \tag{2.8}$$

Thus the magnitude of each term in (2.5) at a given distance from the critical level is not the same above and below it but differs by a factor of $\exp(\pm 2\mu\pi)$, however small c_i may be provided only that it is positive. This is the mathematical ground for the main assertions of this paper. The expressions (2.8), however, oscillate rapidly near $z = z_c$. These results hold for general $N^2(z)$ and $U(z)$ provided these are continuous and sufficiently differentiable. They fail, however, for a situation in which N^2 and U are stepwise constant, and such layered models of a continuous situation are totally inadequate for discussing the present situation.

3. Upward- and downward-propagating waves

In this section we seek to interpret the solutions just described as upward- and downward-travelling waves. In a medium of which the properties vary substantially over a wavelength it is difficult to specify exactly which part of an oscillatory motion corresponds to a wave travelling in the upward direction and which in the opposite one, for there is a continual interchange between the two. In a uniform medium, on the other hand, precise and physically important identifications may be made. When the properties of the medium vary slowly with position such an identification also seems appropriate (Bretherton 1966), and in the present case is helpful in understanding the implications of the mathematics. There are difficulties about imparting precision to it when the value of μ, which is a measure of the Richardson number, is too small, but as our main interest is with large (though not infinite) μ we shall not dwell on them. First, however, we must be quite clear about the corresponding interpretation in a uniform medium.

It is clear from (1.7) that, in a uniform medium in which U and N are constant, every wave with horizontal wave-number k and phase velocity c has a vertical structure of the form

$$\hat{w} = A\,e^{imz} + B\,e^{-imz}, \tag{3.1}$$

where A, B are arbitrary constants, and

$$m = \left\{\frac{N^2}{(U-c)^2} - k^2\right\}^{\frac{1}{2}}. \tag{3.2}$$

For the sake of definiteness, we settle the branch for m by requiring that

$$\text{if} \quad c_i > 0, \quad m_i > 0. \tag{3.3}$$

This implies that

$$\left.\begin{array}{lll} \text{if} & k^2 \ll \dfrac{N^2}{(U-c)^2}, & m \sim +\dfrac{N}{U-c}, \\ \text{and if} & k^2 \gg \dfrac{N^2}{(U-c)^2}, & m \sim ik. \end{array}\right\} \tag{3.4}$$

The complete spatial distribution of velocity associated with the first solution in (3.1) is

$$w = R[A\exp\{i(kx + mz - kct)\}],$$

and it represents a plane wave with phase front

$$kx + mz - kct = \text{constant},$$

at least for the range of frequency kc for which m and c are real. If $U-c$ is negative, so that the wave is propagating in the positive Ox direction relative to the air in which it is, m is also negative, so the phase fronts move downwards, whereas, if $U-c$ is positive, m is also positive. Thus the first solution describes a wave with a downward component of phase velocity. Nevertheless, the influence of such a wave propagates upwards, and we shall interpret this solution as an upward-travelling wave. The second solution

$$\hat{w} = B\,e^{-imz}$$

will likewise be interpreted as a downward-travelling wave. Such an interpretation has been given by Eliassen & Palm (1960) and other writers, but its significance has not always been understood by everyone working in the field, and controversy has resulted (Scorer 1954). We shall look at it in three ways.

In the first, we note that the frequency ω ($=kc$) is given from (3.2) as

$$\omega = kU \pm \frac{kN}{(k^2+m^2)^{\frac{1}{2}}}. \tag{3.5}$$

According to (3.4) we must take the $-$ sign when m and $U-c$ are positive, and the $+$ sign when they are negative. In either case, for the first solution

$$\frac{\partial \omega}{\partial m} = \mp \frac{kNm}{(k^2+m^2)^{\frac{3}{2}}} = \frac{km(U-c)^3}{N^2}, \tag{3.6}$$

which is always positive. But according to theory which is standard in a uniform medium this is the upward component of group velocity. A slow modulation on a sinusoidal train of this wave-number moves without change of shape with this velocity. In a slowly varying medium the form of the modulation may change, but it still moves essentially with the group velocity (Bretherton 1966).

A second view of the first wave solution of (3.1) comes from considerations of energy. The total mean rate of working by the fluid below any level on the fluid above is $\overline{pw}$, where p is the disturbance pressure, and — denotes an average over a horizontal wavelength or over a period. The equation of horizontal momentum for the disturbance

$$(U-c)\,u_x + \frac{1}{\bar{\rho}}\,p_x = 0,$$

where $\bar{\rho}$ is the mean density, shows that

$$+\overline{pw} = -\bar{\rho}(U-c)\,\overline{uw}, \tag{3.7}$$

and it is easily shown that for the solution $\exp(imz)$ the latter is positive. Thus wave energy is flowing upwards. For the solution $\exp(-imz)$ the flow of energy is downwards. However, $\bar{\rho}\overline{uw}$ also describes the mean rate of upward transfer of horizontal momentum by the wave, so such a transfer is inseparable from the flow of energy.

The third way of seeing that the first solution of (3.1) represents an upward-travelling wave is by considering c slightly complex. We have seen that, if the wave motion began at a finite time in the past, it is significant to consider a slowly growing wave with $c_i > 0$. Because of (3.3) the solution $A\exp(imz)$ tends exponentially to zero as $z \to \infty$. Thus the wave amplitude at every point increases with time, but at any given time it is smaller for more positive values of z. Changes in amplitude thus move upwards. The solution $B\exp(-imz)$, on the other hand, is exponentially large as $z \to \infty$, and represents a downward-propagating wave.

Each of these three ways of viewing an upward-travelling wave in a uniform medium has its counterpart when applied to the solutions near the critical level which are given by (2.5). The function $(z-z_c)^{\frac{1}{2}+i\mu}$ oscillates rapidly when $z-z_c$ is small, although the amplitude and wave-number change with position. If $z-z_c$ is real and positive

$$(z-z_c)^{\frac{1}{2}+i\mu} = |z-z_c|^{\frac{1}{2}}[\cos\mu\log|z-z_c| + i\sin\mu\log|z-z_c|]$$

so the local wave-number would seem to be

$$m = \frac{\mu}{z-z_c}. \tag{3.8}$$

If μ is large, this and the wave amplitude change relatively little over a vertical distance m^{-1}, so that the wave is locally sinusoidal. Equation (3.8) also describes variation in phase when $z-z_c$ is real and negative, but according to (2.8) the amplitude is different. Now if U_z is positive

$$\frac{\mu}{z-z_c} = \frac{N}{U_z}\frac{(1-\frac{1}{4}R^{-1})^{\frac{1}{2}}}{z-z_c} \sim \frac{N}{U(z)-c}(1-\tfrac{1}{4}R^{-1})^{\frac{1}{2}} \tag{3.9}$$

and comparison with (3.4) shows that if R is large this is almost identical with the corresponding expression for the wave-number of an upward-travelling wave in a uniform medium. Since (3.9) holds on both sides of the critical level, the wave-number also changes sign with $U(z)-c$, consistent with (3.4).

Alternatively, we consider the Reynolds stress and the vertical energy flux for the two solutions (2.10). Following Miles (1961),

$$\begin{aligned} \overline{uw} &= -\frac{1}{4ik}(\hat{w}^*\hat{w}_z - \hat{w}\hat{w}_z^*) \\ &= -\frac{\mu}{2k}\{|A|^2-|B|^2\} \quad \text{when} \quad z > z_c \\ &= +\frac{\mu}{2k}\{|A|^2e^{2\mu\pi}-|B|^2e^{-2\mu\pi}\} \quad \text{when} \quad z < z_c. \end{aligned} \tag{3.10}$$

This is essentially discontinuous across z_c, for each solution separately changes sign as well as magnitude. However, for the first solution $A(z-z_c)^{\frac{1}{2}+i\mu}$, the energy flux

$$-\bar{p}(U-c)\,\overline{uw}$$

is positive both below and above, and for the second solution it is everywhere negative. So the first wave is associated with an upward transfer of energy.

It was pointed out by Eliassen & Palm (1960) that for any stationary sinusoidal pattern of waves in a shear flow the Reynolds stress is independent of height (corresponding to a constant upward flux of horizontal momentum), but the energy flux is not (because there is an interchange of energy between the wave and the mean flow associated with the transfer of momentum up or down the gradient of mean velocity $U(z)$). Their proof of the constancy of the Reynolds stress fails at a critical level, because any exponential growth of the wave (however small) is crucially important there, and the wave pattern cannot be regarded as unchanging in time. If the upward-travelling wave is identified, as here, with the first solution, the Reynolds stress is reduced in magnitude after the wave has passed through the critical level. Below z_c, the sign of the Reynolds stress for this solution is such that the wave gives up its energy to the mean flow as it moves upwards, and the energy flux decreases with height. Above z_c the wave regains part of its energy, but the measure of its regenerative capacity is the Reynolds stress, which is attenuated by a factor $\exp(-2\mu\pi)$. The second solution $B(z-z_c)^{\frac{1}{2}-i\mu}$ describes a downward-travelling wave, with a downward-energy flux and a Reynolds stress which is smaller in magnitude below z_c. Thus it too is attenuated on passage through the critical layer.

Finally, we look at the amplitude of the motion when $c_i > 0$, and the wave is growing in time. Unlike a wave in a uniform medium, the motion is only different from the case $c_i \to 0$ in a limited region above and below the critical layer; in which the frequency relative to the fluid $k(U-c)$ is comparable to or smaller than the growth rate c_i. However, in this region the amplitude of the vertical velocity for the upward-travelling wave passes smoothly between the expression $A|z-z_c|^{\frac{1}{2}}\exp(\mu\pi)$ appropriate well below the critical level to that appropriate above $A|z-z_c|^{\frac{1}{2}}$, which is substantially smaller. In this region the amplitude of the growing wave decreases upwards. Conversely, for the downward-travelling growing wave the amplitude is larger at the top of the transition region than it is at the bottom.

	$\hat{w}(z)$		Relative phase velocity	$\dfrac{\lvert w\rvert}{\lvert z-z_c\rvert^{\frac{1}{2}}}$	$\overline{uw}$	$\overline{pw}$	Propagating
$U_z > 0$	$A\left(z-\dfrac{c}{U_z}\right)^{\frac{1}{2}+i\mu}$	$z > z_c$	↙	$\lvert A\rvert$	$-\dfrac{\mu}{2k}\lvert A\rvert^2$	+	Up
		$z < z_c$	↘	$\lvert A\rvert e^{\mu\pi}$	$+\dfrac{\mu}{2k}\lvert A\rvert^2 e^{2\mu\pi}$	+	
	$B\left(z-\dfrac{c}{U_z}\right)^{\frac{1}{2}-i\mu}$	$z > z_c$	↖	$\lvert B\rvert$	$+\dfrac{\mu}{2k}\lvert B\rvert^2$	−	Down
		$z < z_c$	↗	$\lvert B\rvert e^{-\mu\pi}$	$-\dfrac{\mu}{2k}\lvert B\rvert^2 e^{-2\mu\pi}$	−	
$U_z < 0$	$A\left(z-\dfrac{c}{U_z}\right)^{\frac{1}{2}+i\mu}$	$z > z_c$	↗	$\lvert A\rvert$	$-\dfrac{\mu}{2k}\lvert A\rvert^2$	−	Down
		$z < z_c$	↖	$\lvert A\rvert e^{-\mu\pi}$	$+\dfrac{\mu}{2k}\lvert A\rvert^2 e^{-2\mu\pi}$	−	
	$B\left(z-\dfrac{c}{U_z}\right)^{\frac{1}{2}-i\mu}$	$z > z_c$	↘	$\lvert B\rvert$	$+\dfrac{\mu}{2k}\lvert B\rvert^2$	+	Up
		$z < z_c$	↙	$\lvert B\rvert e^{\mu\pi}$	$-\dfrac{\mu}{2k}\lvert B\rvert^2 e^{2\mu\pi}$	+	

TABLE 1. Summary of the properties of the two solutions near the critical level. $\mu = [(N^2/U_z^2)-\frac{1}{4}]^{\frac{1}{2}}$ is assumed real and positive throughout. The arrow indicates the quadrant in the (x, z)-plane in which the phase velocity relative to the fluid at that level lies. The wave fronts are always perpendicular to the relative phase velocity, with slope $|w|/|u|$. This slope is equal to $k|z-z_c|$ and hence varies with position.

Before leaving the physical interpretation it should be noticed that for the solutions with $c_i = 0$, although the vertical velocity w is small near $z = z_c$,

$$u = R\left[\frac{\hat{w}_z}{ik}e^{ikx}\right]$$

is large there, and varies in magnitude like $(z-z_c)^{-\frac{1}{2}}$. This is consistent with the particle motion becoming more and more nearly horizontal as z_c is approached; the kinetic energy is entirely in the horizontal motion, but the potential energy is still associated with vertical displacements, so the wave frequency tends to zero. The wave energy per unit volume varies as $(z-z_c)^{-1}$, and the shears associated with the wave vary as $(z-z_c)^{-\frac{3}{2}}$. However, if $c_i \neq 0$ all these infinities at z_c disappear. For a growing wave the horizontal velocities have not had time to build up to a very large value. The singularities are characteristic of a stationary wave train which has persisted for an infinite time. When a wave of definite frequency is being systematically and continuously generated elsewhere in the flow by the boundaries or other causes (as in lee waves) this build-up is of great importance (see §§4, 5). For transient motions, on the other hand, with a continuous distribution of Fourier components, each with infinitesimal amplitudes, singularities in the total integrated disturbance may never appear: they are an artifact of the analysis in each component.

4. The time-dependent disturbance above a sinusoidal corrugation

As an illustration of the remarks of the preceding two sections, we consider here in formal detail a specific problem which is in many respects similar to those considered by Case (1961) and Eliassen, Hoiland & Riis (1953). We take N^2 to be independent of height and $U(z)$ as shown in figure 1.

$$\begin{aligned} U(z) &= U'(z-h) \quad (0 < z < 2h) \\ &= U'h \quad (z > 2h). \end{aligned} \tag{4.1}$$

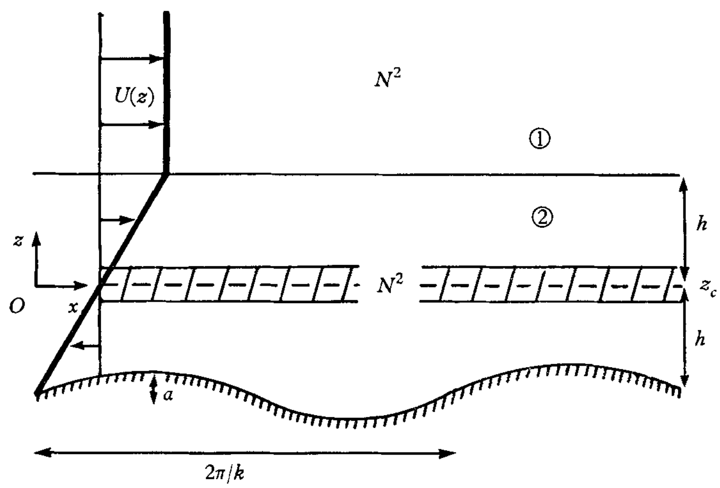

FIGURE 1. The basic state, showing the critical level (broken line) and the critical layer (hatched).

The fluid is unbounded above, and initially at rest everywhere;

$$t < 0; \quad w = 0 \quad \text{everywhere.} \tag{4.2}$$

At time $t = 0$, a disturbance is introduced by raising a sinusoidal corrugation on the lower boundary at $z = 0$, and subsequently maintaining it;

$$t > 0; \quad w = a \cos kx \quad \text{on} \quad z = 0. \tag{4.3}$$

This vertical velocity would be induced by the mean wind $-U'h$ flowing over a corrugation of elevation $-(a/U'hk) \sin kx$. The disturbance due to an isolated mountain would be obtained by superposing such disturbances for many different values of k, but this superposition does not introduce any novel features into the analysis. The assumed time dependence of the elevation of the lower boundary is also unrealistic: it is formally a convenient way of defining a well-posed initial-value problem. An alternative is to envisage the mean flow $U(z)$ increasing from rest to the prescribed profile.

The upper boundary condition is

$$t > 0; \quad w \to 0 \quad \text{as} \quad z \to \infty. \tag{4.4}$$

A long time after the disturbance is introduced, it may be necessary to look at very large values of z to find the disturbance small, but we are assuming that, for any given t, it is always possible. The source of the disturbance is thus unequivocally at $z = 0$.

The governing differential equation for small perturbations $w(x, z, t)$ is equation (2.1). If the amplitude a is sufficiently small, the linearization on which this is based is self-consistent a long time after the motion begins. For any given amplitude, however, it ultimately breaks down. Nevertheless, we investigate the solution up to the time that the theory becomes inconsistent, and may in principle justify this for any value of t by taking a sufficiently small. With the broken line profile of equation (4.1), U_{zz} is everywhere zero, except at the height $z = 2h$. At this level U_{zz} in the governing equation may be replaced by a delta function

$$U_{zz} = -U'\delta(z-2h) \tag{4.5}$$

or equivalently the pressure and vertical velocities may be matched across a perturbed interface between two separate fluids in regions (1) and (2). The latter procedure leads to the same linearized matching conditions at $z = 2h$ as are obtained by formally applying (4.5).

It is convenient at this stage to introduce dimensionless variables

$$\xi = \frac{x}{h}, \quad \zeta = \frac{z-h}{h}, \quad \tau = U't, \tag{4.6}$$

a dimensionless wave-number and phase velocity

$$\kappa = kh, \quad \gamma = c/U'h, \tag{4.7}$$

where, without loss of generality, κ is taken as positive; together with the Richardson number

$$R = \mu^2 + \tfrac{1}{4} = N^2/U'^2. \tag{4.8}$$

To obtain a complete solution, we assume everywhere a sinusoidal variation with ξ, and then Laplace transform in time,

$$w(\xi, \zeta, \tau) = R[\tilde{w}(\zeta, \tau)\, e^{i\kappa\xi}]$$

$$\hat{w}(\zeta, \gamma) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \tilde{w}(\zeta, \tau)\, e^{i\kappa\gamma\tau}\, d\tau, \tag{4.9}$$

bearing in mind that the relevant part of the complex plane is where $\gamma_i > 0$. This integral (4.9) then converges rapidly for all disturbances which do not increase exponentially or faster as $\tau \to \infty$. The Laplace transform of the governing equation assumes different forms in regions (1) and (2):

$$\text{region (1):} \quad \hat{w}_{\zeta\zeta} + \left\{\frac{R^2}{(1-\gamma)^2} - \kappa^2\right\}\hat{w} = 0 \quad \text{in} \quad \zeta > 1, \tag{4.10}$$

$$\text{region (2):} \quad \hat{w}_{\zeta\zeta} + \left\{\frac{R^2}{(\zeta-\gamma)^2} - \kappa^2\right\}\hat{w} = 0 \quad \text{in} \quad -1 < \zeta < 1. \tag{4.11}$$

Each of these is simply Scorer's equation (1.7). The curvature of the velocity profile is not neglected; it appears in the matching conditions

$$\left.\begin{aligned} \hat{w}_1 &= \hat{w}_2, \\ \hat{w}_{1\zeta} - \hat{w}_{2\zeta} + \frac{1}{1-\gamma}\hat{w} &= 0 \end{aligned}\right\} \quad \text{at} \quad \zeta = 1. \tag{4.12}$$

The remaining boundary conditions are

$$\text{region (1):} \quad \hat{w} \to 0 \quad \text{as} \quad \zeta \to \infty, \tag{4.13}$$

$$\text{region (2):} \quad \hat{w} = -\frac{a}{\surd(2\pi)}\frac{1}{i\kappa\gamma} \quad \text{on} \quad \zeta = -1. \tag{4.14}$$

The pole at $\gamma = 0$ in (4.14) arises from the specific time dependence assumed in (4.3). If the forcing at $z = 0$ is removed again after a finite time τ_0, the factor $1/i\kappa\gamma$ must be replaced by

$$\frac{1}{i\kappa\gamma}(1 - e^{-i\kappa\gamma\tau_0}), \tag{4.15}$$

which has no singularities in the complex γ plane. The singularities in the steady solution of our problem which appear as $\tau \to \infty$ are associated with this pole, and are a consequence of the forcing being maintained for an indefinitely long time.

In region (1), the only solution of equation (4.10) which is consistent with the upper boundary condition (4.13) is

$$\hat{w} = A_1 e^{m_1(\zeta-1)}, \tag{4.16}$$

where

$$m_1 = \left\{\frac{R^2}{(1-\gamma)^2} - \kappa^2\right\}^{\frac{1}{2}} \tag{4.17}$$

with $\mathscr{I}(m_1) > 0$ when $\gamma_i > 0$. This branch is forced by the vanishing of w for large ζ under the conditions for which the Laplace transform (4.9) is convergent, i.e. $\gamma_i > 0$.

In region (2), the general solution is

$$\hat{w} = (\zeta-\gamma)^{\frac{1}{2}}\{A_2 I_{i\mu}(\kappa\zeta - \kappa\gamma) + B_2 I_{-i\mu}(\kappa\zeta - \kappa\gamma)\}, \tag{4.18}$$

where $I_{i\mu}$, $I_{-i\mu}$ are the modified Bessel functions of the first kind, with complex order $\pm i\mu$. Near the origin

$$I_{i\mu}(\lambda) \sim \lambda^{i\mu}\frac{(\frac{1}{2})^{i\mu}}{\Gamma(i\mu+1)},$$

so, if the constant A is replaced by

$$A_2(-\tfrac{1}{2}\kappa)^{i\mu}/\Gamma(i\mu+1),$$

the structure of equation (4.18) near $\zeta = \gamma$ is identical with that described in equation (2.9).

It is a straightforward matter of algebra to apply conditions (4.12) at the interface and (4.14) at the lower boundary to determine the constants A_1, A_2, B_2. We write, for brevity,

$$I'_{i\mu}(\lambda) = (d/d\lambda)\, I_{i\mu}(\lambda),$$

$$\lambda_1 = \kappa(1-\gamma), \quad \lambda_2 = \kappa(-1-\gamma),$$

and

$$Q(\gamma) = \{\tfrac{1}{2} + im_1(1-\gamma)\}\{I_{i\mu}(\lambda_2)\, I_{-i\mu}(\lambda_1) - I_{-i\mu}(\lambda_2)\, I_{i\mu}(\lambda_1)\} + \lambda_1\{I'_{i\mu}(\lambda_1)\, I_{i\mu}(\lambda_2) - I'_{-i\mu}(\lambda_1)\, I_{i\mu}(\lambda_2)\}.$$

Then in region (1)

$$\hat{w} = \frac{a}{\sqrt{(2\pi)}}\frac{2i\mu}{i\kappa\gamma}\frac{1}{Q}\left(\frac{1-\gamma}{-1-\gamma}\right)^{\frac{1}{2}} e^{im_1\zeta}, \tag{4.19}$$

whereas in region (2) $\hat{w}$ is determined by

$$A_2 = \frac{a}{\sqrt{(2\pi)}}\frac{1}{i\kappa\gamma}\frac{\{\frac{1}{2} + im(1-\gamma)\}\, I_{-i\mu}(\lambda_1) - \lambda_1 I'_{-i\mu}}{Q}\frac{1}{(-1-\gamma)^{\frac{1}{2}}}, \tag{4.20}$$

$$B_2 = \frac{a}{\sqrt{(2\pi)}}\frac{1}{i\kappa\gamma}\frac{\{\frac{1}{2} + im(1-\gamma)\}\, I_{i\mu}(\lambda_1) - \lambda_1 I'_{i\mu}}{Q}\frac{1}{(-1-\gamma)^{\frac{1}{2}}}. \tag{4.21}$$

The complete formal solution to the problem is then given by the inverse Laplace transform

$$\hat{w}(x,z,t) = R\left[e^{i\kappa\xi}\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty\,\Gamma}^{+\infty} \hat{w}(\gamma,\zeta)\, e^{-i\kappa\gamma\tau}\kappa\, d\gamma\right], \tag{4.22}$$

where the contour of integration Γ lies along the real γ-axis, except where there is a singularity in the integrand, in which case it lies above, in $\gamma_i > 0$.

5. Asymptotic analysis for large τ

The integral (4.22) is in general hopelessly complicated, but if τ is large methods of asymptotic analysis akin to that of 'steepest descents' (Jeffreys & Jeffreys 1946, p. 472) may be applied to give great simplifications. The dominant contributions to the integral come from neighbourhoods in the complex γ plane of points where either the integrand is singular, or the derivative with respect to γ of the coefficient of τ in the exponent vanishes. If ζ is kept finite as $\tau \to \infty$, there are no points in the latter category ('saddle points') and the largest contribution comes from the pole at $\gamma = 0$.

We shall see that this implies that, except in a neighbourhood of the critical level at $\zeta = 0$ which shrinks with time, the motion everywhere becomes that for a standing wavepattern satisfying Scorer's equation, except that the magnitude of the motion above the critical layer is given by the matching conditions developed in §2. Thus, if the Richardson number ~ 1 or greater, the motion above the layer is very drastically reduced in magnitude. Superposed on this steady pattern are several small decaying oscillations. One of these is the remnant of the transient waves induced by the impulsive start to the motion as they are absorbed in the shear layer, each at the critical level appropriate to their frequency.

There is a region above and below the critical level which decreases in thickness as time goes on and in which the motion is not yet steady, even to a first approximation. We shall call this region the critical layer. Above and below it the steady state is achieved quite quickly; within it the maximum magnitude of the horizontal velocities increases with time, but after any finite interval the velocities and their spatial derivatives are everywhere finite and well behaved. In the critical layer the horizontal momentum associated with the upward-travelling wave is nearly all transferred into the mean flow and the wave is effectively absorbed.

If ζ/τ is kept fixed as $\tau \to \infty$, the largest contribution comes from a saddle point. It describes an upward-moving dispersing group of waves, the dominant frequency at any point being exactly such that the corresponding vertical component of group velocity is ζ/τ. Above this group the disturbance has not yet penetrated; below it the steady-state solution of Scorer's equation is achieved. It describes the influence of the impulsive start to the motion, but ultimately passes by any given point.

To see these results it is necessary to catalogue the singularities of the integrand in equation (4.22). These are:

(*a*) $\gamma = 0$, pole, arising from the applied boundary condition (4.3);

(*b*) $\gamma = -1$, branch point, $\lambda_2 = 0$;

(*c*) $\gamma = +1$, branch point plus essential singularity, $\lambda_1 = 0$, $m_1 \to \infty$;

(*d*) $\gamma = 1 \pm R/\kappa$, branch point, $m_1 = 0$;

(*e*) $\gamma = \zeta$, $|\zeta| < 1$, branch point, $(\zeta-\gamma)^{\frac{1}{2}} I_{\pm i\mu}[\kappa(\zeta-\gamma)] = 0$.

In addition there are possible poles where $Q = 0$. It is difficult to be certain where all these are. There are none in $\gamma_i > 0$. This follows from a theorem proved by Howard (1961), that in a general stratified shear flow there are no exponentially growing or decaying normal modes for a disturbance which vanishes on the boundaries and at infinity, provided the Richardson number is everywhere larger than $\frac{1}{4}$. A zero of Q in $c_i > 0$ corresponds precisely to such an exponentially growing mode. Howard's theorem does not rule out zero's in $\gamma_i < 0$, for the expression for $\hat{w}$, continued analytically across the real axis from $\gamma_i > 0$, does not vanish as $z \to \infty$. Such zeros need not concern us here, as they describe exponentially decaying solutions. Since $I_{i\mu}$ is essentially a complex number even if its argument is real, it seems unlikely that there are any on the real γ axis. If attention is restricted to λ_1, λ_2 small (equivalent to considering only waves whose vertical scale is small compared with their horizontal scale), the zeros of Q may be found explicitly and are not relevant to the present analysis, although in placing them care must be taken to remain on the correct Riemann sheet in the γ-plane, corresponding to a continuous deformation from $\gamma_i > 0$, with the branch of $I_{\pm i\mu}$ correctly defined.

Assuming that singularity (*e*) does not coincide with any of the others, we deform the contour of integration Γ according to figure 2, so that almost all of it lies in the region $\gamma_i < 0$. As $\tau \to \infty$ the integrand is exponentially small, except in those regions which are near the real axis $\gamma_i = 0$.

For fixed ζ, the dominant contribution to the integral (4.19) comes from the pole at $\gamma = 0$, and is equal to $2\pi i$ times the residue at the pole. Thus, as $\tau \to \infty$,

$$w(\xi, \zeta, \tau) \to R[(2\pi)^{\frac{1}{2}} i\kappa \lim_{\gamma\to 0} (\gamma \hat{w}(\gamma))\, e^{i\kappa\xi}]. \tag{5.1}$$

In region (1)
$$w \sim R\left[-\frac{2\mu a}{Q(0)} \exp\left[i(R^2-\kappa^2)^{\frac{1}{2}}\zeta + i\kappa\xi\right]\right],$$

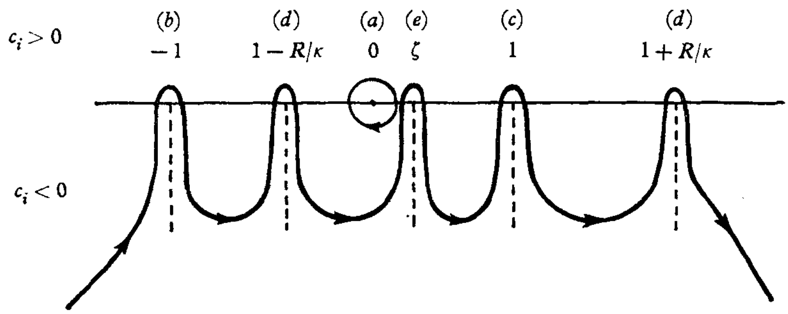

FIGURE 2. The contour of integration, after deformation.

which is a stationary upward-propagating wave as interpreted in §3. In region (2)

$$w \sim R[e^{i\kappa\xi}\zeta^{\frac{1}{2}}\{A_2'(0)\, I_{i\mu}(\kappa\zeta) + B_2'(0)\, I_{-i\mu}(\kappa\zeta)\}],$$

where
$$A_2'(0) = \lim_{\gamma\to 0} (\gamma A_2(\gamma)). \tag{5.2}$$

The first term describes an upward-propagating wave, but the magnitude of $I_{i\mu}$ is, for given $|\zeta|$, smaller by a factor $\exp(-\mu\pi)$ above the critical level $\zeta = 0$ than it is below it. The critical level thus exercises a crucial influence on the flow. If μ is large (> 1) there is very little motion anywhere above the critical layer.

If ζ/τ is held fixed and positive as $\tau \to \infty$, there is a saddle point for the exponent in (4.19), where

$$\frac{d}{d\gamma}\left(m_1 \frac{\zeta}{\tau} - \kappa\gamma\right) = 0, \tag{5.3}$$

i.e. where
$$\frac{1}{\kappa}\frac{d}{d\gamma}\left\{\frac{R^2}{(1-\gamma)^2} - \kappa^2\right\}^{\frac{1}{2}} = \frac{\tau}{\zeta}.$$

This is exactly the condition that the vertical component of group velocity of a wave of frequency $\kappa\gamma$ in the uniform region (1) should equal ζ/τ, and the standard method of steepest descents yields the contribution to the integral, which describes a wave of frequency equal to that at the saddle point, which decreases in amplitude like $\tau^{-\frac{1}{2}}$. This decrease is consistent with conservation of wave energy because the band of frequencies corresponding to the region between two slightly different values of ζ/τ occupies a larger and larger region of physical space as $\tau \to \infty$.

The contributions to the integral associated with the remaining singularities all tend to zero as $\tau \to \infty$ for fixed ζ. This follows from the Riemann–Lebesgue lemma, because in the neighbourhood of each, along the real axis for γ,

$$\int_\alpha^\beta |\hat{w}(\gamma)|\, dy < \infty$$

for any two points α, β on the real axis near the singularity. Then, according to the lemma

$$\int_\alpha^\beta \hat{w}(\gamma)\, e^{-i\kappa\gamma\tau} d\gamma \to 0 \quad \text{as} \quad \tau \to \infty.$$

The same is also true for the first vertical derivative of w, so (5.1) describes the dominant part of the velocity field as $\tau \to \infty$ for fixed ζ ($\neq 0, \pm 1$).

However, it is of interest to examine these contributions more closely, particularly from singularity (e).

The contribution from the neighbourhood of $\gamma = \zeta$ (singularity (e)) may be evaluated approximately by putting

$$\gamma = \zeta + \delta, \tag{5.4}$$

and expanding the integrand as a power series in δ, on the assumption that only those values of δ which are small are of interest. The leading term of the series in region (2) has the form

$$e^{-i\kappa\zeta\tau} \int_{\Gamma_e} (-\delta/2\pi)^{\frac{1}{2}} \{A_2(\zeta)(-\delta)^{i\mu} + B_2(\zeta)(-\delta)^{-i\mu}\} e^{-i\kappa\delta\tau} \kappa\, d\delta, \tag{5.5}$$

where $A_2(\zeta)$, $B_2(\zeta)$ are the values of A_2, B_2 when γ is put equal to ζ, and the contour of integration Γ_e is that portion of the deformed contour which lies near the point $\gamma = \zeta$. It passes from $\delta_i < 0$ to $\delta_i < 0$ round the branch point at $\delta = 0$. The rapid convergence of the integral (5.5) when δ_i becomes negative is ensured by the factor $\exp(-i\kappa\delta\tau)$. If τ is large only small values of δ contribute to the integral, and ignoring all but the leading term of the power series expansion is justified. It may be written

$$w = R\left[\kappa \exp[i\kappa(\xi - \zeta\tau)] \left\{\frac{A_2(\zeta)C(\mu)}{(\kappa\tau)^{\frac{3}{2}+i\mu}} + \frac{B_2(\zeta)C(-\mu)}{(\kappa\tau)^{\frac{3}{2}-i\mu}}\right\}\right], \tag{5.6}$$

where

$$C(\mu) = \left\{\int_0^\infty \theta^{\frac{1}{2}+i\mu} e^{-\theta} d\theta\right\} \{\exp[3\pi(\mu+i)/4] - \exp[-\pi(\mu+i)/4]\}. \tag{5.7}$$

The main interest of equation (5.6) lies in its structure. It describes locally plane sinusoidal waves with phase $\kappa(\xi - \zeta\tau)$. The lines of constant phase are thus everywhere advected with the basic flow velocity ζ, and tilted to a nearly horizontal orientation as $\tau \to \infty$. The horizontal wave-number remains constant, but the vertical wave-number becomes large.

The amplitude of the vertical velocity at any given height decays as $\tau^{-\frac{3}{2}}$, but the horizontal velocity (which is proportional to w_ζ) decreases only as $\tau^{-\frac{1}{2}}$. The plane waves are also modulated by the amplitude functions $A_2(\zeta)$, $B_2(\zeta)$ and the time variation $\tau^{\mp i\mu}$, but these variations are slow compared with the changes in phase.

It is easily verified that the functions

$$w = R[e^{ik(x-Ut)} t^{-\frac{3}{2} \mp i\mu}] \tag{5.8}$$

are solutions of the basic governing equation (2.1) in an unbounded uniform shear $U = U'z$, provided the motion is quasi-horizontal ($\partial^2/\partial x^2 \ll \partial^2/\partial z^2$). The energy density for such motions is uniform and decays as t^{-1}; the Reynolds stress is also uniform, but decays as t^{-2}. The Reynolds stress for both upward- and downward-travelling solutions has the same sign, the advected wavefronts of the plane waves being orientated so that wave energy is transformed to the mean flow. Equation (5.7) shows that the Reynolds stress for the upward-travelling part is dominated by the motion for each component below its appropriate critical level, and conversely for the downward-travelling part.

The oscillation is thus being absorbed into the mean flow. This absorption is associated with the continuous distribution of the disturbance over a band of frequencies, each frequency with a distinct critical level. The resulting disturbance has no discontinuities in velocity, although the shear (proportional to w_{zz}) does increase with time.

In the velocity field above a sinusoidal corrugation, these decaying plane waves are the final remnant of the waves of all frequencies generated at the impulsive start. There are other remnants, associated with the branch points (*b*), (*c*) and (*d*). The structure of these can be investigated in a similar way; they all describe oscillations with the same frequency all over the flow field, and all decay to zero as $\tau \to \infty$. These singularities are a consequence of the broken line velocity profile we adopted, and probably would not appear if a more realistic variation of $U(z)$ were assumed. However, this will not be pursued here.

So far we have assumed that none of the singularities in the list (*a*)–(*e*) coincide with one another. Thus the points $\zeta = -1, 0$ or $+1$ have been excluded from the discussion. When singularity (*e*) coincides with the points (*b*), (*c*) or (*d*), the integrand is still absolutely integrable and the corresponding contribution tends to zero as $\tau \to \infty$. More interesting, however, is the region $\zeta \to 0$ as $\tau \to \infty$, i.e. within the critical layer. The separation of the contributions from singularities (*a*) and (*e*) is then no longer permissible; their neighbourhoods must be treated together and a different asymptotic expression for the contribution is required. It may be found by expanding the integrand as a power series in $\gamma - \zeta$ but assuming ζ is also small. If we write

$$\gamma - \zeta = \lambda/\kappa\tau,$$

the leading term in the series is

$$w \sim R\left[\frac{e^{i\kappa\xi}}{(\kappa\tau)^{\frac{1}{2}+i\mu}}\frac{A_2'(0)\,\kappa}{(2\pi)^{\frac{1}{2}}}\int_{\Gamma_{ae}}\frac{(-\lambda)^{\frac{1}{2}+i\mu}}{\lambda+\kappa\tau\zeta}e^{-i\lambda}d\lambda + \frac{e^{i\kappa\xi}}{(\kappa\tau)^{\frac{1}{2}-i\mu}}\frac{B_2'(0)\,\kappa}{(2\pi)^{\frac{1}{2}}}\int_{\Gamma_{ae}}\frac{(-\lambda)^{\frac{1}{2}-i\mu}}{\lambda+\kappa\tau\zeta}e^{-i\lambda}d\lambda\right]. \quad (5.9)$$

The contour Γ_{ae} is from $\lambda = -i\infty$ to $\lambda = -i\infty$ round both $\lambda = 0$ and $\lambda = -\kappa\tau\zeta$.

Unfortunately, even these simplified integrals cannot be integrated in terms of elementary functions, but the structure of the solution is clear. Each integral is a function of the parameter $\kappa\tau\zeta$. If this is large two separate contributions from the pole and the branch point are appropriate, as previously described. If $\kappa\tau\zeta$ is of order unity, however, w and its vertical derivatives are smoothly varying functions of ζ and τ, but the velocity field is essentially time dependent. The width of this critical layer in which, even a very long time after starting, the flow has still not settled down to a steady state is comparable to $(\kappa\tau)^{-1}$, and thus decreases with time. A typical vertical velocity in it has magnitude of order $\tau^{\frac{1}{2}}$; a typical horizontal velocity is of order $\tau^{-\frac{1}{2}}$. Thus ultimately non-linear terms which have been ignored to date will become important, and the theory becomes invalid. However, in principle this invalidation may be delayed indefinitely by taking a small enough. The important result is that, long before this, the flow above and below the critical layer has settled down to a steady state, with a small decaying oscillation superimposed, and that steady state is the same as that obtained by ignoring the details of the critical layer, but matching round it in the way described in § 2. Above and below the critical layer the Reynolds stress is independent of height, but it is much smaller above it. The change in the Reynolds stress, which is associated with continuing absorption of momentum from the wave, takes place across the critical layer, and the resulting acceleration of the mean flow is confined within an ever-narrowing band of heights. The sign of this acceleration of the mean flow is such as to decrease the height of the critical layer with time.

Finally, it is worth remarking on the simplifications which occur when $\kappa \ll 1$, and the motion is everywhere very nearly horizontal. Then λ_1, λ_2 are small for the final steady state $\gamma = 0$, and $m_1 \sim R = (\mu^2+\frac{1}{4})^{\frac{1}{2}}$. Also

$$I_{i\mu}(\lambda) \sim \frac{(\frac{1}{2})^{i\mu}}{\Gamma(i\mu+1)}\lambda^{i\mu}$$

throughout the region of interest $-\kappa < \lambda < \kappa$. Then

$$Q(0) \sim \frac{1}{|\Gamma(i\mu+1)|^2}\{(\tfrac{1}{2}+i(\mu^2+\tfrac{1}{4})^{\frac{1}{2}})(e^{\mu\pi}-e^{-\mu\pi})+i\mu(e^{\mu\pi}+e^{-\mu\pi})\}. \tag{5.10}$$

If μ is greater than about unity, this is dominated by the factor $e^{\mu\pi}$, compared to which $e^{-\mu\pi}$ is negligible. Under such circumstances

$$\left.\begin{aligned} A_2' &\sim \frac{a}{\sqrt{(2\pi)}\,\kappa}\Gamma(i\mu+1),\\ B_2' &\sim \frac{a}{\sqrt{(2\pi)}\,\kappa}\Gamma(-i\mu+1)\frac{\frac{1}{2}+i(\mu^2+\frac{1}{4})^{\frac{1}{2}}-i\mu}{\frac{1}{2}+i(\mu^2+\frac{1}{4})^{\frac{1}{2}}+i\mu}.\end{aligned}\right\} \tag{5.11}$$

In the region (2) above the critical layer the ratio of the vertical energy fluxes or of the Reynolds stresses associated with the downward- and upward-travelling waves is

$$\frac{|B_2'|^2}{|A_2'|^2} = \frac{(\mu^2+\frac{1}{4})^{\frac{1}{2}}-\mu}{(\mu^2+\frac{1}{4})^{\frac{1}{2}}+\mu} \sim \frac{1}{16\mu^2}. \tag{5.12}$$

Thus very little energy is reflected by the discontinuity in U_z at the interface between regions (1) and (2). Below the critical layer the discrepancy between the energy fluxes is even larger, being

$$\frac{|B_2'|^2}{|A_2'|^2}e^{-2\mu\pi} \sim \frac{1}{16\mu^2}e^{-2\mu\pi}.$$

Thus, if $\mu > 1$, the effect of the region above the critical layer on the region below it is quite negligible; the critical layer acts as an absorbing barrier of great effectiveness.

6. Transient disturbances in a shear flow

A similar analysis to that of §4 may be used to describe the disturbance due to a transient stimulus in a shear layer, with or without layers of uniform velocity above and below. Unfortunately the resulting integral which is the formal solution is hopelessly intractable. Nevertheless, it is possible to make some general statements about what happens.

First, we consider the asymptotic perturbation velocity field a long time after a spatially sinusoidal stimulus. In the analysis there is now no pole at $\gamma = 0$, so the velocities everywhere decay with time, and the largest contributions come from singularities of types (*b*)–(*e*) defined in §5. Of these, (*b*), (*c*) and (*d*) are artifacts of the chosen broken line basic velocity profile, and would not be present in an unbounded uniform shear flow. They are associated with decaying oscillations which are coherent over the whole flow field.

Of more interest are the velocities associated with singularity (*e*). It was shown in §5 that these have the form

$$w = \mathscr{R}\left[\frac{F(z)}{t^{\frac{3}{2}+i\mu}} + \frac{G(z)}{t^{\frac{3}{2}-i\mu}}\right] e^{ik(x-U(z)t)}. \tag{6.1}$$

Each term describes locally plane waves of very small vertical wavelength (the vertical wave-number is $-kt\,dU/dz$), and with a frequency given by simple advection by the local mean flow of a pattern which is periodic in x. The amplitude of the vertical velocity decays like $t^{-\frac{3}{2}}$, but the horizontal perturbation velocity is given by

$$u = \mathscr{R}\left[\left(\frac{F(z)}{t^{\frac{1}{2}+i\mu}} + \frac{G(z)}{t^{\frac{1}{2}-i\mu}}\right) U_z e^{ik(x-Ut)}\right] \tag{6.2}$$

and decays more slowly. Advected decaying plane wave solutions of this type have also been invoked by Phillips (1966) in a proposed energy spectrum for internal waves.

It is clear from their derivation that the decay of velocity fields of the form (6.1) is a manifestation of critical-layer absorption for a continuous spectrum of frequencies. Each frequency is associated with a critical level z_c, and at each height z there is a corresponding frequency kc for which it is critical. The function $F(z)$ is proportional to the amplitude in upward-travelling waves of that frequency excited by the original disturbance. It provides a spatial display of the frequency spectrum.

A qualitative explanation of this effect is provided by the following argument. If μ is large, the development of the original disturbance may be followed using concepts of wave packets and group velocity. It was shown by Bretherton (1966) that a wave packet travelling vertically should stagnate in a region near the critical level corresponding to its dominant frequency. After a long time a disturbance composed of wave packets of all frequencies but initially without a clear spatial structure should be redistributed by the dispersive wave propagation until the dominant frequency at each level is that for which the vertical component of group velocity is vanishingly small, i.e. for which that level is critical.

It should also be noticed that equation (6.1) is concerned only with disturbances which are purely sinusoidal in x. In practice a stimulus is localized in space as well as time, and it is necessary to integrate equation (6.1) over a continuous spectrum of wave-numbers k. Explicit formulae cannot be obtained for a general initial disturbance, but, if the latter can be represented as a slowly varying modulation $A(x)$ on a non-zero dominant wave-number k_0, it appears to be consistent to integrate with respect to x after the asymptotic form for large time has been obtained. If attention is concentrated on a fixed value of z, the horizontal dependence of the disturbance is then entirely accounted for by replacing the factor $\exp ik\{x - U(z)t\}$ in equation (6.1) by

$$A(x - Ut)\, e^{ik_0(x-Ut)} \tag{6.3}$$

and k by k_0 wherever else it implicitly occurs. The contribution of this form is associated entirely with the continuous spectrum of horizontal phase velocities c for which a critical level lies within the fluid. In addition there may be contributions which disperse in a horizontal direction from propagating components without a critical level. These, however, inevitably propagate to infinity faster than the advection speed $U(z)$ at any level. Thus a wave packet clustered around a dominant wave-number k_0 does not behave qualitatively differently from a single wave-number, and the superposition of a continuum of wave-numbers seems not to introduce any surprising novel features.

The absence of horizontal dispersion at each level z, which is implied by expression (6.3), is consistent with the propagation of wave packets. It was shown (Bretherton 1966) that, when a packet moves vertically towards its critical level z_c, both the vertical component of group velocity and the difference from $U(z_c)$ of the horizontal component decrease to zero as t^{-2}. The displacement of the packet in both directions relative to a frame of reference moving horizontally with velocity $U(z_c)$ is thus strictly bounded, and all packets, wherever they originate, which have the same critical level soon tend to be moving together without dispersion. If a packet is originally moving in a vertical direction away from its critical level, it either propagates away completely or is internally reflected within a finite time, in which case it again ends up near level z_c.

It is now possible to check the validity of the linearization on which the whole calculation is based (§2, assumption (*e*)). Using velocity fields (6.1) and (6.2) we may estimate the non-linear terms

$$uw_x + ww_z = O(F^2 t^{-2}), \tag{6.4}$$

whereas

$$w_t + Uw_x = O(Ft^{-\frac{5}{2}}). \tag{6.5}$$

Thus, in the absence of dissipation, the non-linearities eventually become important after a time of order F^{-2}, however small F may be, and the linearization fails. At the same time the velocity gradients u_z have become comparable to the basic shear U_z, so it is possible that the flow becomes turbulent. However, the vertical scale of the perturbations is then of order $k_0^{-1}F^2$, so any turbulence which results may be expected to be of low intensity. Also, by this time the disturbance energy is already almost completely absorbed into the mean flow, so the main conclusions of this paper should be unaffected.

A final result which may be obtained simply concerns the total change of momentum of the mean flow as a result of the absorption of the disturbance. To discuss this we write the total velocity as

$$U(z)+U_1(x,z,t)+u(x,z,t), \quad W_1(x,z,t)+w(x,z,t). \tag{6.6}$$

Here the perturbations u, w and U_1, W_1 are assumed to be localized in space, vanishing sufficiently rapidly at all finite times for their integrals to converge. U_1, W_1 may roughly be described as second-order Eulerian mean velocities, associated with first-order small-amplitude disturbances u, w of zero mean. However, their separation from u, w need be more precise only to the extent of specifying

$$\int_{-\infty}^{+\infty} u\,dx = \int_{-\infty}^{+\infty} w\,dx = 0. \tag{6.7}$$

It then follows from continuity that, if W_1 vanishes at some level,

$$\int_{-\infty}^{+\infty} W_1\,dx = 0.$$

Now

$$\int_{-\infty}^{+\infty} U_1\,dx$$

does not necessarily vanish, and may be identified as the mean flow momentum at that level associated with the disturbance, and we shall calculate its total change during the absorption process.

The equation for the horizontal velocities is

$$\frac{\partial}{\partial t}(U+U_1+u)+\frac{\partial}{\partial x}(U+U_1+u)^2+\frac{\partial}{\partial z}\{(U+U_1+u)(W_1+w)\}+\frac{1}{\rho}p_x = 0. \tag{6.8}$$

Integrating with respect to x, and remembering that p, u, w, U_1 are assumed to vanish sufficiently rapidly as $|x|\rightarrow\infty$,

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} U_1\,dx = -\frac{\partial}{\partial z}\int_{-\infty}^{+\infty}(U_1+u)(W_1+w)\,dx. \tag{6.9}$$

If the disturbance to the basic flow $U(z)$ is of small amplitude, it is certainly consistent to assume that $U_1 \ll u$, $W_1 \ll w$, and to calculate w to first order in the manner of the remainder of this paper. Thus we have

$$\left[\int_{-\infty}^{+\infty} U_1\,dx\right]_{t=0}^{\infty} = -\int_0^{\infty} dt \int_{-\infty}^{+\infty} dx\,\frac{\partial}{\partial z}(uw). \tag{6.10}$$

This is a familiar result concerning the role of the Reynolds stress. It is important to realize that it involves only first-order theory, although computation of U_1 at each point involves solving second-order equations.

Expressing the disturbance as a double integral over a continuum of wave-numbers and horizontal phase velocities according to equations (2.4) and (2.6), we have from Parseval's theorem

$$\int_0^{\infty} dt \int_{-\infty}^{+\infty} dx\,uw = \int_{-\infty}^{+\infty} k\,dc \int_0^{\infty} dk\,\overline{uw}, \tag{6.11}$$

where

$$\overline{uw} = \frac{1}{4ik}(\hat{w}\hat{w}_z^* - \hat{w}^*\hat{w}_z) \tag{6.12}$$

is the Reynolds stress for each Fourier component as computed in §3. Thus the total momentum transferred upwards past each level is simply the integral over all wave-numbers and frequencies of the Reynolds stress associated with each Fourier component separately. This is only true of the total effect, not at any finite time.

Now the Reynolds stress for each Fourier component is independent of height above and below its critical level, but is discontinuous there, so the only contributions to the right-hand side of equation (6.10) at any given height z come from those for which

$$c = U(z). \tag{6.13}$$

Integration with respect to c is then equivalent to integration with respect to z, and

$$\left[-\int_{-\infty}^{+\infty} U_1\, dx\right]_{t=0}^{\infty} = -U_z \int_0^{+\infty} dk\, \overline{uw}\Big]_{z-}^{z+}, \tag{6.14}$$

where $\overline{uw}]_{z-}^{z+}$ is the discontinuity in Reynolds stress associated with that Fourier component with phase velocity defined by equation (6.13) and wave-number k.

Equation (6.14) is the main result of this section. It states that the total transfer of momentum to the mean flow associated with the passage and partial absorption of a transient disturbance is finite, distributed over a range of heights, and calculated from the discontinuity across the critical layer (if any) of each Fourier component separately. If the disturbance is initiated below $z = 0$ and travels upwards, and if the Richardson number is moderately large, the upward-travelling wave is almost completely absorbed at its critical level. The discontinuity is then simply minus the value of the Reynolds stress at $z = 0$, which is in turn by equation (3.7) directly connected to the net upward flux of wave energy at $z = 0$. It has been derived only for a localized disturbance, but it is clear that the result can be generalized to a spatially homogeneous distribution of random disturbances. The integral on the left of equation (6.14) must be replaced by the mean horizontal velocity at a point, and the Reynolds stress for each Fourier component replaced by the average per unit distance in the Ox direction.

7. Conclusions

Internal gravity waves propagating with a vertical component of group velocity in shear flow are almost completely absorbed at a critical level (if it exists) at which the horizontal component of phase velocity of the wave is equal to the mean velocity of the fluid normal to the wave front, provided the Richardson number R_c there is larger than about 1. For $\frac{1}{4} < R_c < 1$ less complete absorption occurs, measured by the transmission coefficient $\exp\{-2\pi(R_c-\frac{1}{4})^{\frac{1}{2}}\}$ for the energy flux divided by the local relative frequency. The case $R_c < \frac{1}{4}$ is not discussed. This absorption is associated with a change in the Reynolds stress between two constant values above and below the critical layer, implying a transfer of horizontal momentum by the wave into the mean flow around the critical level. Previous authors (Eliassen & Palm 1960) have shown that, even in the absence of a critical level, there is a distributed interchange of wave energy with the mean flow at all levels, but, for a sinusoidal wave with real phase velocity, the Reynolds stress should be independent of height, except possibly at a critical level.

The absorption mechanism is not dependent on viscosity or other dissipative processes, and its modification by these has not yet been investigated. It arises out of a linear theory of small perturbations on the basic flow, an ambiguity in the solution being settled by consideration of an initial-value problem in which the motion at some given time is quite general but assumed known and allowed to develop subsequently, rather than being restricted by the requirement that at a later time the velocity field must conform to an arbitrary and predetermined specification. A more physical picture has been given by Bretherton (1966), in which it is shown that a wave packet travelling with the appropriate local group velocity will approach the critical level but never reach it, being neither transmitted nor reflected. Near the critical level the vertical wavelength becomes very small. However, the analysis of this paper covers a wider range of circumstances, in which the concepts of wave packet and group velocity cannot be clearly defined, for which the Richardson number is not necessarily large, nor is the vertical scale of variation of the Brunt–Väisälä frequency and of the vorticity of the basic flow much greater than the vertical wavelength for the waves.

It is, however, required that the distribution of $U(z)$, $V(z)$ is continuous and differentiable at least near any critical levels. A model of a shear flow in which the mean velocity is uniform in discrete layers, as considered by Hines & Reddy (1966), cannot give for a pure sinusoidal component the partial transmission or the change in Reynolds stress which are obtained here, for the critical level is inevitably either buried in the zone of discontinuity of mean velocity between two layers where the local Richardson number is effectively zero, or distributed over a complete layer in which the relative phase velocity is zero and in which infinitesimal wave theory is quite inconsistent. Indeed, the conclusions reached by Hines & Reddy (1966) are significantly different from the present ones: in the absence of viscosity they claim total *reflexion* at the critical level, with no absorption of momentum, and rely on viscosity to dissipate the wave there, reducing the process to one of *absorption*.

The analysis of §§4, 5 shows that, if a single Fourier component describing a sinusoidal wave travelling in the horizontal direction is excited continuously by some exterior mechanism, the horizontal velocities in a layer of decreasing thickness about the critical level will, according to linearized inviscid theory, increase in magnitude systematically and indefinitely, though less rapidly than indicated by conservation of wave energy. Thus ultimately non-linearities become important however small the forcing mechanism and it seems plausible that a breakdown to turbulence will occur. However, the magnitude of the transmitted wave is established long before this occurs, apparently independently of the details of the flow in the critical layer.

Furthermore, if the forcing is transient and only active for a finite time the indefinite build-up of the horizontal velocities does not occur. Such a transient disturbance can only be described by a continuous band of Fourier components, each with its own critical level, and the critical level absorption takes place over a range of heights. Equation (5.6) describes the asymptotic velocity distribution a long time later where the last remnants of the disturbance appear as locally plane waves advected by the shear and decaying in magnitude. Even if linear theory does eventually break down, for any disturbance which is started from rest the absorption of momentum by the mean flow and the partial transmission of the wave are largely complete before the non-linearities have grown substantial. In §6 it is shown that the total time-integrated transfer of momentum to the mean flow by the Reynolds stress for a localized transient disturbance may be computed from the Reynolds stress at one level for each Fourier component separately.

The main geophysical applications of these conclusions which appear at present are in the propagation of gravity waves in the atmosphere from the troposphere to the ionosphere, and in the transfer of horizontal momentum by internal waves from the surface down to the interior of the ocean, without associated mixing of water, salt and heat. The first of these is described in Bretherton (1966) and developed further in an independent study by Hines & Reddy (1966). The second is still unexplored, but potentially significant. A third example on which it throws some light is concerned with lee waves, when the incident stream over the obstacle reverses at some height. It is predicted here that there should be little disturbance above the critical level, but large-amplitude perturbations of the horizontal velocity should build up near there, and turbulent breakdown seems probable. Gerbier & Berenger (1961) report turbulence found by glider pilots at such heights, with little or no lee waves above, but their findings are not obviously consistent with the case studies by which they are illustrated and the evidence is somewhat obscure.

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